Days in Logic 2022

Universidade do Algarve

June 30 – July 2, 2022

https://daysinlogic2022.ualg.pt/

Welcome Address

Days in Logic is a biennial meeting aiming at bringing together logicians, mathematicians, computer scientists and other scientists from Portugal (but also elsewhere) with interest in Logic. It is specially directed to graduate students. Previous editions were held in Braga (2004), Coimbra (2006), Lisboa (2008), Porto (2010), Évora (2012), Braga (2014), Monte da Caparica (2016), Aveiro (2018) and Lisboa (2020).

The 10th edition of Days in Logic is a hybrid event taking place at the University of Algarve, Faro, from the 30th of June to the 2nd of July 2022.

The programme consists of three tutorials by invited speakers and twelve contributed talks.

The previous edition of Days in Logic occurred just before the outbreak of the current pandemic. This year, the meeting has for the first time a hybrid format, allowing in-person attendees and remote participants.

It is our pleasure to welcome you all to Faro for three enjoyable "days in logic"!

The Organizing Committee
Daniel Graça, José Espírito Santo, Gilda Ferreira

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Contents

Invited speakers - Tutorials

Weihrauch Complexity

Vasco Brattka
Universität der Bundeswehr München

Weihrauch complexity is now an established and active part of mathematical logic. It can be seen as a computability-theoretic approach to classifying the uniform computational content of mathematical problems. This theory has become an important interface between more proof-theoretic and more computability-theoretic studies in the realm of reverse mathematics. In this tutorial we plan to present an introduction into Weihrauch complexity and several facets of this theory. We hope to address most of the following topics:

- An intuitive introduction into Weihrauch complexity with examples.
- The way the Weihrauch lattice can be seen as a general calculus for mathematical problems.
- The role of choice principles within this theory.
- The relation of these principles to axiom systems in reverse mathematics, as well as to computational classes such as computability with finitely many mind changes, limit computability, Las Vegas computability etc.
- The algebraic structure of the Weihrauch lattice.
- Its relation to linear and intuitionistic logic.
- The way Weihrauch complexity refines Borel complexity.
- Concrete examples of classifications, such as Ramsey's theorem, Brouwer's fixed point theorem etc.
- The way certain structural properties of the lattice might depend on the underlying axiomatic setting (ZFC versus ZF+AD).

The following handbook chapter covers most of the presented topics:

Brattka, V. and Gherardi, G. and Pauly, A., Weihrauch Complexity in Computable Analysis, in: *Handbook of Computability and Complexity in Analysis*, Springer, Cham, 2021, pages 367-417

(Boolean) Satisfiability and its Applications

Mikoláš Janota
Czech Technical University in Prague

SAT solvers are computer programs that automatically try to decide whether a given Boolean formula is satisfiable or not. Despite the problem being NP-complete, modern SAT solvers are able to solve problems with millions of variables. This enables us to use them to tackle interesting combinatorial problems.

In this tutorial we will look at:

1. How problems from other domains can be modeled as Boolean satisfiability.
2. Which techniques make SAT solvers so powerful.
3. How can SAT be used to solve more expressive logics, pertaining to the domain of satisfiability modulo theories (SMT).

Univalent Combinatorics

Egbert Rijke
University of Ljubljana

I will give an introduction to univalent mathematics, and illustrate it with applications of the univalence axiom in combinatorics. Unlike Zermelo-Fraenkel set theory, univalent mathematics is an extension of Martin-Lof's dependent type theory with the univalence axiom and propositional truncations, and often further extensions are considered as well. The univalence axiom asserts that identifications of types are equivalently described as equivalences of types. A feature of univalent mathematics is that all constructions in it are automatically equivalence invariant. Furthermore, when you define a concept in univalent type theory, you automatically define the type of all objects in that concept. In other words, by defining the concept of group, we get the type of all groups; by defining the concept of n-gon we get the type of all n-gons; by defining the concept of cube, we get the type of all cubes, and so on. In each of these cases, the univalence axiom ensures that the identity type of the type of all objects of a given concept are the isomorphisms or symmetries of that object. We will see that this leads to a natural way of doing group theory via pointed connected 1-types. Furthermore, we will see how the univalence axiom along with some basic homotopy theory helps with the general problem of counting up to isomorphism.

Contributed talks

Computational properties of PNmatrices

Carlos Caleiro, Pedro Filipe, Sérgio Marcelino
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Instituto Superior Técnico, Universidade de Lisboa (Portugal)

Partial non-deterministic matrices (PNmatrices) were introduced in the beginning of this century [1, 2, 3] as a generalization of logical matrices, by allowing the connectives to be functionally interpreted as partial multi-functions, rather than functions. The added expressiveness allows for finite characterizations of a much wider class of logics and general recipes for various problems in logic, such as procedures to constructively update semantics when imposing new axioms [5, 6], or effectively combining semantics for two logics, capturing the effect of joining their axiomatizations [4, 8]. Whenever the underlying logic is expressive enough, PNmatrices also allow for general techniques for effectively producing analytic calculi for the induced logics, over which a series of reasoning activities in a purely symbolic fashion can be performed, including proof-search and countermodel generation.

Many problems regarding logics induced by finite matrices are known to be decidable. These include, for given matrices, checking: whether the induced logic has theorems, or is expressive enough, and also if the induced logics have the same set of theorems, or are simply equal. These problems had not been studied in the wider context of PNmatrices and, in this talk, we provide answers to some of them. The landscape is quite rich, as some problems keep their computational status, other increase in complexity, and for a few, decidability is lost [7, 9]. Some of the results are obtained by exploring connections between PNmatrices, termDAG automata and counter machines.

Acknowledgments Research funded by FCT/MCTES through national funds and when applicable co-funded by EU under the project UIDB/50008/2020.

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Formalising Approximation Fixpoint Theory

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Approximation Fixpoint Theory (AFT) is an abstract lattice-theoretic framework originally designed to unify semantics of non-monotonic logics [1]. Its first applications were on unifying all major semantics of logic programming, autoepistemic logic (AEL), and default logic (DL), thereby resolving a longstanding issue about the relationship between AEL and DL [2, 3]. AFT builds on Tarski's fixpoint theory of monotone operators on a complete lattice, starting from the key realisation that, by moving from the original lattice L to the bilattice L^2, Tarski's theory can be generalized into a fixpoint theory for arbitrary (i.e., also non-monotone) operators. Crucially, all that is required to apply AFT to a formalism and obtain several semantics is to define an appropriate approximating operator $L^2 \to L^2$ on this bilattice; the algebraic theory of AFT then directly defines different types of fixpoints that correspond to different types of semantics of the application domain.

In the last decade, AFT has seen several new application domains, including abstract argumentation, extensions of logic programming, extensions of autoepistemic logic, and active integrity constraints. Around the same time, also the theory of AFT has been extended significantly with new types of fixpoints, and results on *stratification, predicate introduction*, and *strong equivalence*. All of these results were developed in the highly general setting of lattice theory, making them directly applicable to all application domains, and such ensuring that researchers do not "reinvent the wheel".

Given the success and wide range of applicability of AFT, it sounded natural to formalise this theory in the Coq theorem prover. In this work we give a short introduction to AFT and the challenges of this formalisation.

Acknowledgments This work was partially supported by Fonds Wetenschappelijk Onderzoek – Vlaanderen (project G0B2221N).

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The Halpern-Mann iteration in CAT(0) spaces

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Complete CAT(0) spaces, also known as Hadamard spaces, are a non-linear generalization of Hilbert spaces. Benefiting from ideas and tools from the proof mining program (see e.g. [3]) it was shown in [1] the strong convergence of an iterative schema which alternates between Halpern and Krasnoselskii-Mann style iterations in the general context of CAT(0) spaces. At the same time, the logical tools used allowed to obtain quantitative information in the form of rates of asymptotic regularity and rates of metastability (in the sense of T. Tao). If one restricts oneself to Hilbert spaces, the proof follows some standard arguments. However, to obtain the proof in the more general context of CAT(0) spaces the use of logical tools, and in particular the technique introduced in [2], seem to be necessary. In this talk I will explain the role of the logical tools in obtaining this result.

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Formal Theories of Occurrences

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Whereas syntactic entities (as terms, formulae and formula trees) are treated carefully in logic and related fields, this is not true anymore with respect to their *occurrences*. A brief survey of the literature yields a great variety of intuitive and/or inaccurate approaches; the best definition is found in computer science (for example, Huet [2]), but the proposed theory of occurrences is too weak to capture the concept of occurrences in its full strength. In particular, central concepts of logic, as, for example, the discharge functions for Natural Deduction derivations are not adequately definable in these weak theories.

This unsatisfactory situation was the motivation to investigate in our dissertation [1] the notion of occurrences in some details. Thereby, an occurrence is determined by three aspects: an occurrences is always an occurrence *of* a syntactic entity (its *shape*) *in* a syntactic entity (its *context*) at a specific *position* (the critical aspect).

Context and shape are meaningful combinations of the underlying well-known syntactic entities. Our crucial idea was to represent the positions by so called *nominal forms*, essentially as introduced by Schütte [3]; as a consequence, we obtained a canonical, strong and general theory of occurrences.

In the first part of our talk, we provide a more philosophically oriented introduction into the notion of occurrences; basic intuitions and concepts are established, the relevance of the notion of occurrences is illustrated and a brief historical survey of the treatment of occurrences in the literature is given.

The second part of our talk focuses on the formal theories of occurrences developed in our dissertation, more specifically on the formal theory of occurrences of terms in terms of a first order language. This paradigmatic theory is sufficiently complex to investigate occurrences in their full generality and is easily carried over to arbitrary types of occurrences.

Once having introduced the basic concepts (including formal occurrences of different degrees of generality) and the fundamental methods for their treatment, we show how to deal with more elaborate concepts related with (formal) occurrences, as their *identity*, the *lies-within* relation and the *independence* of occurrences. If time permits, we conclude our talk by a brief discussion of operations on (independent) occurrences.

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Decidability of combined logics, and applications

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Transference theorems have always been a main drive of the research in combined logics. Decidability is certainly one of the most desirable properties a logic should have, opening the way for the development of tool support for logical reasoning. Capitalizing on the insights brought by recent modularity results regarding propositional many-valued logics [5, 6], we present very general sufficient conditions, based on syntactic extensibility criteria, assuring that the combination of two decidable logics is still decidable, and establish upper bounds for the complexity of the corresponding decision problem. We show that the proposed technique covers many important cases, namely when the logics being combined have disjoint signatures [4], as well as the fusion of modal logics [3]. Furthermore, we shall see how the proposed criteria smoothly adapt to applications beyond the propositional-based case, including the combination of equational and first-order theories [1, 2].

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Two pure ecumenical natural deduction systems

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Natural deduction systems, as proposed by Gentzen [1] and further studied by Prawitz [4], is one of the most well known proof-theoretical frameworks. Part of its success is based on the fact that natural deduction rules present a simple characterization of logical constants, especially in the case of intuitionistic logic. However, there has been a lot of criticism on extensions of the intuitionistic set of rules in order to deal with classical logic. Indeed, most of such extensions add, to the usual introduction and elimination rules, extra rules governing negation. As a consequence, several meta-logical properties, the most prominent one being *harmony*, are lost.

In [5], Dag Prawitz proposed a natural deduction *ecumenical system*, where classical logic and intuitionistic logic are codified in the same system. In this system, the classical logician and the intuitionistic logician would share the universal quantifier, conjunction, negation and the constant for the absurd, but they would each have their own existential quantifier, disjunction and implication, with different meanings. Prawitz' main idea is that these different meanings are given by a semantical framework that can be accepted by both parties.

In this talk, we propose two different approaches adapting, to the natural deduction framework, (a) Girard's mechanism of *stoup* [2] and (b) Murzi's proposal [3] of combining Peter Schröder-Heister's higher-level rules [6] and Neil Tennant's idea of the $\bot$ as a punctuation sign [7]. This will allow the definition of a pure harmonic natural deduction system ($\mathcal{LE}_p$) for the propositional fragment of Prawitz' ecumenical logic [5].

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Modal embeddings and calling paradigms

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The first connection between modal embeddings and calling paradigms in functional programming was made in the context of linear logic [3, 4]. In our work, we seek the root and the deep meaning of this connection, along the following general lines first laid out in [1]: (1) One can identify a modal calculus (a simple extension of the λ-calculus with an S4 modality) that serves as target of the modal embeddings and show that this modal target obeys a new calling paradigm, named call-by-box; (2) The embeddings of intuitionistic logic into modal logic S4 attributed to Girard and Gödel [6] can be recast as maps compiling respectively the ordinary (call-by-name, cbn) λ-calculus and Plotkin's call-by-value (cbv) λ-calculus [5] into the modal target, achieving together a unification of call-by-name and call-by-value through call-by-box; (3) One can define later instantiations of the S4 modality, in the form of interpretations of the modal target into diverse other calculi, recovering by composition known embeddings like, for instance, the ones into the linear λ-calculus [4].

In the context of linear logic [3, 4], Gödel's embedding has slightly weaker properties than Girard's. In our work [1], such an asymmetry remained: For cbn, the treatment is so neat that we may say Girard's embedding just points out an isomorphic copy of the cbn λ-calculus as a fragment of the modal target calculus. For cbv and Gödel's embedding, the results were not so satisfying. In a recent work [2], we show that, by refining the modal target calculus and accordingly recasting the embeddings, we do not lose the neat treatment of cbn, and obtain similar results for cbv, that is: Gödel's embedding becomes just the indication of an isomorphic copy of Plotkin's cbv λ-calculus as a fragment of the modal target. In this sense, the ordinary and Plotkin's λ-calculi truly co-exist inside a simple modal calculus.

In this talk we intend to recall the general lines of the work laid out in [1] and to present the recent improvements obtained in [2].

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The Halpern-Mann iteration in UCW-hyperbolic spaces

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Recently, an iterative method for approximating common fixed points of two nonexpansive maps was introduced. This method alternates between the well-known Halpern-type and Mann-type definitions, and was thus dubbed the Halpern-Mann iteration (HM). In [1], the HM method was studied in the setting of CAT(0) spaces, and asymptotic regularity and strong convergence were established. UCW-hyperbolic spaces were introduced in [2] as uniformly convex W-hyperbolic spaces where a monotone modulus of uniform convexity is available, and are a natural generalization of both uniformly convex normed spaces and CAT(0)-spaces.

In this talk, we will see that it is possible to extend the asymptotic regularity result to the setting of UCW-hyperbolic spaces. We shall discuss why such result points to the nature of the iterative scheme HM being a proper mixing of the Halpern and the Mann algorithms. Finally, we shall look at a particular choice of parameters where linear rates of convergence are available. This is ongoing joint work with Laurenţiu Leuştean [3], set in the context of the proof mining program.

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The Arithmetical Nature of the Hydra Game

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Mathematically speaking, a hydra is a finite rooted tree whose top-nodes are called the heads of the hydra. The so-called Hydra Game, as it was introduced by Laurence Kirby and Jeff Paris in [4], stages a fight between a hydra and an Herculean opponent whose goal is to defeat the hydra by cutting off the totality of its heads.

The game develops by alternating opponent's attacks and hydra's replies. The opponent invariably proceeds by chopping off exactly one of the hydra's heads at a time. Hydra's counterattack depends on whether the head chopped off is directly connected to the root-node (its body) or not. Suppose the opponent decides to chop off a head h lying at level n of a hydra $\mathcal{T}$, i.e., a top-node located n edges away from $\mathcal{T}$'s root:

- If $n > 1$, then the hydra fights back by regrowing from h's grandfather-node h' (the node connected with h and located at level $n-2$) an *arbitrarily large* number of copies of its mutilated subtree (whose root-node is in fact h');

- If $n = 1$ (the head h is directly connected to the root-node) the hydra is not allowed to regenerate itself and just forfeits its turn.

The game goes on till the opponent is finally able to reduce the hydra to a single-point body without heads, i.e., to the single-point graph.

We call *strategy* any function telling the opponent which head to chop off at each stage of a certain game. The well-known theorem proved by Paris and Kirby establishes that:

> [KP] whatever strategy the opponent decides to follow, it turns out to be successful, namely able to defeat the hydra after a finite (possibly enormously large) number of moves [4].

Put in this way, the specific strategy the opponent decides to implement may only affect the number of steps needed to reach the end of the game, not the final outcome which is always against the hydra.

Kirby and Paris' demonstration is notoriously led by transfinite induction. In a nutshell, the argument describes a way to decorate each node of any given hydra with an ordinal number. Such a decoration recursively proceeds from the heads to the root-node so that the ordinal attached to the root is finally taken to measure the size of the whole hydra. The bottom line of the proof consists in showing that sizes decrease at each stage of any game, therefore, sooner or later, any sequence of attacks and counterattacks will hit the single-node tree having size 1.

Once the term 'strategy' is taken as referring to a specific class of recursive functions, KP can be formalized in Peano Arithmetic (PA). What makes really interesting a property that would otherwise be just a mere combinatorial curiosity, is the fact that KP proves to be independent of PA [4,2].

As a first contribution, we offer a new proof for KP which is led only by means of combinatorial principles, without mentioning ordinals and transfinite induction [3]. The second contribution focuses on the role KP plays in the foundations of mathematics. In particular, we stress Abe's encoding function [1] to develop a rewriting system completely isomorphic to the Hydra Game, yet fully number-theoretic in nature. This move allows us to point out a new Goodstein-like sentence independent of PA.

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Adding abductive reasoning to a propositional logic

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We present a technique for obtaining a logic with abductive reasoning extending a given propositional logic. Abduction, along with deduction and induction, is recognized as important for machine learning, namely in identifying possible causes that may lead to the occurrence of an event and in providing new ways for a computational device to achieve a certain objective. Each rule in the original calculus induces a set of multiple-conclusion abductive rules. Moreover, rules stating generic properties of abduction have to be added. In the induced logic, the deductive mechanism of the base logic coexists with this abductive component. A new notion of a multiple-conclusion derivation had to be developed. Due to the canonical nature of obtaining such a logic, we prove the preservation of soundness, completeness, decidability, and computational complexity. These concepts and results are illustrated in a robot navigation problem using a multimodal logic. The talk is based on the paper [1].

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'Provability Implies Provable Provability' in FLINSPACE

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We create a theory of arithmetic for the class $\mathsf{FLINSPACE}$ (this class is the same as Grzegorczyk's class $\mathcal{E}^2$) that we call G_2. Our approaches are distinct from what is found in the literature, in particular the theory that we develop for $\mathsf{FLINSPACE}$ includes $\mathsf{I}\Delta_0$.

We explore connections between G_2 and $\mathsf{I}\Delta_0$, and their implications in the study of complexity classes. After that, we express the usual meta-mathematical notions in G_2: we define numerations of the axioms of a theory; we define the *standard proof predicate* $\mathsf{Prf}_\xi(x, y)$ that expresses "y is the code of a proof of the formula coded by x according to the numeration ξ"; and we define the *standard provability predicate* $\mathsf{Pr}_\xi(x) := \exists y.\mathsf{Prf}_\xi(x, y)$.

It is a known fact that the derivability condition "provability implies provable provability" is very sensitive to the considered theory: more precisely, we have no guarantee that it holds for weak theories of arithmetic (it is an open problem for $\mathsf{I}\Delta_0$).

We study the uniform derivability condition $\mathsf{Pr}_\xi(x) \to \mathsf{Pr}_\xi(\ulcorner \mathsf{Pr}_\xi(\dot{x}) \urcorner)$. We prove that if $\mathsf{Pr}^{\mathcal{S}}(x)$ is a provability predicate for a finite set of axioms $\mathcal{S}$ (including a finite number of logical axioms), then $\mathsf{G}_2 \vdash \mathsf{Pr}^{\mathcal{S}}(x) \to \mathsf{Pr}_\xi(\ulcorner \mathsf{Pr}_\xi(\dot{x}) \urcorner)$. Moreover, if G_2 can verify its axioms, in the sense that, for a suitable G_2-function $\mathsf{verifier}$,

$\mathsf{G}_2 \vdash \xi(x) \to \mathsf{Prf}_\xi(\ulcorner \mathsf{Pr}_\xi(\dot{x})\urcorner, \mathsf{verifier}(x))$, then $\mathsf{G}_2 \vdash \mathsf{Pr}_\xi(x) \to \mathsf{Pr}_\xi(\ulcorner \mathsf{Pr}_\xi(\dot{x})\urcorner)$. A sufficient condition for the internal Σ_1-completeness of G_2 is also presented. Finally, we present conditions for a numeration ξ_0 of a finitely axiomatizable theory to satisfy $\mathsf{G}_2 \vdash \mathsf{Pr}_{\xi_0}(x) \to \mathsf{Pr}_\xi(\ulcorner \mathsf{Pr}_{\xi_0}(\dot{x})\urcorner)$.

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Decidability of Consequence via Reduction

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We adopt consequence systems as the right framework for introducing logical (decision) problems such as the Consequence Problem, the Theoremhood Problem and the Consistency Problem. Reductions play a key role in the reflection and preservation of decidability and non decidability, respectively. We consider two levels of reduction: reduction between decision problems and between consequence systems. Then, we establish sufficient conditions for the existence of reductions between problems over the same consequence system as well as across different consequence systems. We also analyze the relationship between the two levels of reductions. Several examples are presented covering paraconsistent, intuitionistic, and modal logics.

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